KUWAIT UNIVERSITY
Department of Mathematics & Computer Science

Math 102 | Summer 2005
Calculus B | Final Exam | Time: 2 hours

Use of calculators is not allowed in this exam. Please switch off your mobile phones.

1. Use logarithmic differentiation to find $\dfrac{dy}{dx}$, where [3 pts]

$$y = \frac{x^{\ln \cosh x}\, 3^{\sin^{-1}\sqrt{x}}}{\sqrt[3]{1-2x}}$$

2. Simplify $\tanh\left(\ln \dfrac{e}{x}\right)$. [3 pts]

3. Find $\displaystyle\lim_{x\to 1^+} (\ln x)^{x-1}$ if it exists. [3 pts]

4. Evaluate the following integrals: [3 pts each]

a. $\displaystyle\int \frac{dx}{x\sqrt{x^3-1}}$

b $\displaystyle\int \frac{dx}{\sqrt{1+\sqrt[3]{x}}}$

c. $\displaystyle\int \frac{dx}{(4x-x^2)^{3/2}}$

d. $\displaystyle\int \frac{\ln 2}{2^x - 3\,(2^{-x}) + 2}\,dx$

5. Determine whether the following integral is convergent or divergent. [4 pts]

$$\int_0^1 \ln(x+x^2)\,dx$$

6. Sketch the cardioid $r = 2 - 2\sin\theta$ and the circle $r = 2\sin\theta$ and label their points of Intersection with the axes. [2 pts]

a. Show that the cardioid has horizontal tangent lines at $(1, \pi/6)$ and $(1, 5\pi/6)$. [2 pts]

b. Find the arc length of the part of the cardioid corresponding to $0 \le \theta \le \pi/2$. [3 pts]

c. Find the area of the region that is inside the circle and outside the cardioid. [4 pts]

7. Identify and sketch the following conic section: [4 pts]

$$25x^2 + 4y^2 + 50x - 16y - 59 = 0$$

Find its focus (foci), vertex (vertices) and its directrix or asymptotes based on its type.

Q1 $$y = \frac{x^{\ln\cosh x} \cdot 3^{\sin^{-1}\sqrt{x}}}{\sqrt[3]{1-2x}}$$ taking ln both sides

$$\ln y = \ln\cosh x \ln x + \sin^{-1}\sqrt{x} \ln 3 - \frac{1}{3}\ln(1-2x)$$

Differentiating w.r.t 'x'

$$\frac{1}{y}\frac{dy}{dx} = \ln\cosh x \cdot \frac{1}{x} + \ln x \cdot \frac{1}{\cosh x}\sinh x + \frac{1}{\sqrt{1-x}} \cdot \frac{1}{2}x^{-\frac{1}{2}} \ln 3 - \frac{1}{3(1-2x)}(-2)$$

$$\frac{dy}{dx} = \left(\frac{\ln\cosh x}{x} + \tanh x \ln x + \frac{\ln 3}{2\sqrt{x}\sqrt{1-x}} + \frac{2}{3(1-2x)}\right)y$$

$$\therefore \frac{dy}{dx} = \left[\frac{\ln\cosh x}{x} + \tanh x \ln x + \frac{\ln 3}{2\sqrt{x-x^2}} + \frac{2}{3(1-2x)}\right]\frac{x^{\ln\cosh x} \cdot 3^{\sin^{-1}\sqrt{x}}}{\sqrt[3]{1-2x}}$$

Ans

Q2 $$\tanh\left(\ln\frac{e}{x}\right) = \tanh(\ln e - \ln x) = \tanh(1-\ln x)$$

$$= \frac{e^{1-\ln x} - e^{-1+\ln x}}{e^{1-\ln x} + e^{-1+\ln x}} = \frac{e^{1} \cdot e^{\ln x^{-1}} - e^{-1} \cdot e^{\ln x}}{e^{1} \cdot e^{\ln x^{-1}} + e^{-1} \cdot e^{\ln x}} = \frac{\frac{e}{x} - \frac{x}{e}}{\frac{e}{x} + \frac{x}{e}}$$

$$= \frac{e^2 - x^2}{e^2 + x^2}$$ Ans

Q3 $$\lim_{x \to 1^+} (\ln x)^{x-1} \qquad 0^0$$

let $y = \lim_{x\to 1^+} (\ln x)^{x-1}$, $\therefore \ln y = \lim_{x\to 1^+} (x-1)\ln(\ln x)$

$$= \lim_{x\to 1^+} \frac{\ln(\ln x)}{\frac{1}{x-1}} \qquad \frac{\infty}{\infty}$$

$$= \lim_{x\to 1^+} \frac{\frac{1}{x\ln x}}{-\left(\frac{1}{x-1}\right)^2} = -\lim_{x\to 1^+} \frac{(x-1)^2}{x\ln x} \qquad \frac{0}{0}$$

$$= \lim_{x\to 1^+} \frac{2(x-1)}{1+\ln x} = \frac{0}{1} = 0$$

$\therefore \ln y = 0 \qquad \therefore y = e^0 = 1$

$$\therefore \lim_{x\to 1^+} (\ln x)^{x-1} = 1$$ Ans.

Q4 (a) $\int \frac{1}{x\sqrt{x^3-1}}\,dx$ let $x^3-1=u^2$, $3x^2dx = 2u\,du$

$= \frac{1}{3}\int \frac{3x^2}{x^3\sqrt{x^3-1}}\,dx = \frac{1}{3}\int \frac{2\cancel{u}\,du}{(u^2+1)\cancel{u}} = \frac{2}{3}\tan^{-1}u + C$

$= \frac{2}{3}\tan^{-1}\sqrt{x^3-1} + C$

b) $\int \frac{dx}{\sqrt{1+\sqrt[3]{x}}}$ let $u^2 = 1+\sqrt[3]{x} \Rightarrow u^2-1 = \sqrt[3]{x}$

$\therefore x = (u^2-1)^3$, $\therefore dx = 3(u^2-1)^2 \cdot 2u\,du = 6u(u^2-1)^2 d$

$\int \frac{6\cancel{u}(u^2-1)^2du}{\cancel{u}} = 6\int (u^4-2u^2+1)\,du = 6\left[\frac{u^5}{5} - 2\frac{u^3}{3} + u\right]$

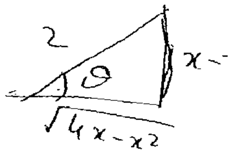
$= 6\left[\frac{(1+\sqrt[3]{x})^{5/2}}{5} - 2\frac{(1+\sqrt[3]{x})^{3/2}}{3} + \sqrt{1+\sqrt[3]{x}}\right] + C$

c) $\int \frac{1}{(4x-x^2)^{3/2}}\,dx$ $\Rightarrow 4x-x^2 = -(x^2-4x+4-4)$
$= 2^2-(x-2)^2$

$= \int \frac{1}{[2^2-(x-2)^2]^{3/2}}\,dx$ let $x-2 = 2\sin\theta$
$dx = 2\cos\theta\,d\theta$

$\int \frac{1}{[4-4\sin^2\theta]^{3/2}}\,2\cos\theta\,d\theta = \int \frac{1}{8\cos^3\theta}\cdot 2\cos\theta\,d\theta$

$= \frac{1}{4}\int \sec^2\theta\,d\theta = \frac{1}{4}\tan\theta + C$



$= \frac{1}{4}\cdot\frac{x-2}{\sqrt{4x-x^2}} + C$

d) $\int \frac{\ln 2}{2^x - 3\cdot 2^{-x} + 2}\,dx = \int \frac{2^x \ln 2\,dx}{(2^x)^2 + 2\cdot 2^x - 3}$ let $2^x = u$, $2^x\ln 2\,dx = du$

$= \int \frac{1}{u^2+2u-3}\,du = \int \frac{1}{(u+3)(u-1)}\,du = \frac{1}{4}\int\left(\frac{1}{u-1} - \frac{1}{u+3}\right)du$

$= \frac{1}{4}\ln\left|\frac{u-1}{u+3}\right| + C = \frac{1}{4}\ln\left|\frac{2^x-1}{2^x+3}\right| + C$

Q5 $\int_0^1 \ln(x+x^2)\,dx = \lim_{t\to 0^+}\int_t^1 \ln(x+x^2)\,dx = \lim_{t\to 0^+}\left[x\ln(x+x^2) - \int \frac{x+2x^2}{x+x^2}\right.$

$= \lim_{t\to 0^+}\left[x\ln(x+x^2) - \int \frac{1+2x}{1+x}\,dx\right]_t^1 = \lim_{t\to 0^+}\left[x\ln(x+x^2) - \int\left(2-\frac{1}{1+x}\right)\right]_t^1$

$= \lim_{t\to 0^+}\left[x\ln(x+x^2) - 2x + \ln(1+x)\right]_t^1 = (\ln 2 - 2 + \ln 2) - \lim_{t\to 0^+}\left[t\ln(t+t^2) - 2t + \ldots\right]$

$= 2\ln 2 - 2 - \left[\lim_{t\to 0^+} t\ln(t+t^2)\right] = 2\ln 2 - 2$

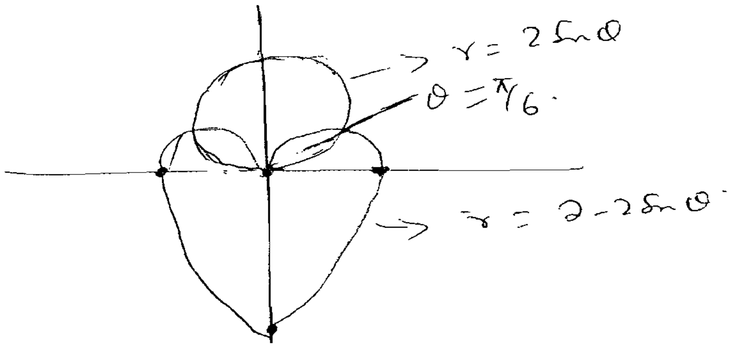
As lim [illegible]

Conv

Q6 $r = 2 - 2\sin\theta$ $r = 2\sin\theta$ ③

θ	$-\pi/2$	0	$\pi/2$
r	4	2	0

θ	$-\pi/2$	0	$\pi/2$
r	-2	0	2



a) As $m = \dfrac{dy}{dx} = \dfrac{r\cos\theta + \sin\theta \frac{dr}{d\theta}}{-r\sin\theta + \cos\theta \, dr/d\theta}$

for Horizontal tangent line $r\cos\theta + \sin\theta \frac{dr}{d\theta} = 0$.

$\Rightarrow (2 - 2\sin\theta)\cos\theta + \sin\theta(-2\cos\theta) = 0$

$2\cos\theta - 2\sin\theta\cos\theta - 2\sin\theta\cos\theta = 0$

$\cos\theta - 2\sin\theta\cos\theta = 0 \Rightarrow \cos\theta[1 - 2\sin\theta] = 0$

$\cos\theta = 0$ $1 - 2\sin\theta = 0 \Rightarrow \sin\theta = \frac{1}{2}$

$\theta = \frac{\pi}{6}, \frac{5\pi}{6}$.

i.e. Horizontal tangent line pts are $(1, \frac{\pi}{6}), (1, \frac{5\pi}{6})$

b) Arc length of Polar equation $= \int_0^{\pi/2} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \, d\theta$

$\therefore = \int_0^{\pi/2} \sqrt{4(1-\sin\theta)^2 + 4\cos^2\theta} \cdot d\theta$

$= 2\int_0^{\pi/2} \sqrt{1 - 2\sin\theta + \sin^2\theta + \cos^2\theta} \, d\theta = 2\int_0^{\pi/2} \sqrt{2 - 2\sin\theta} \, d\theta$

$= 2\sqrt{2}\int_0^{\pi/2} \sqrt{1 - \sin\theta} \, d\theta = 2\sqrt{2}\int_0 \left(-\sin\frac{\theta}{2} + \cos\frac{\theta}{2}\right) d\theta$

$= 2\sqrt{2} \left| +2\cos\frac{\theta}{2} + 2\sin\frac{\theta}{2} \right|_0^{\pi/2}$

$= 2\sqrt{2}\left[\left(+2 \cdot \frac{1}{\sqrt{2}} + 2 \cdot \frac{1}{\sqrt{2}}\right) - (2)\right]$

$= 2\sqrt{2}\left[+\frac{4}{\sqrt{2}} \mp 2\right] = 2\sqrt{2}\,[2\sqrt{2} - 2)$

$=$ $= 1$

c) Point of Intersection

$2\sin\theta = 2 - 2\sin\theta \Rightarrow 4\sin\theta = 2 \Rightarrow \sin\theta = \frac{1}{2}, \quad \theta = \frac{\pi}{6}$

$$A = \cancel{8}\cdot\frac{1}{2}\int_{\pi/6}^{\pi/2}\left[4\sin^2\theta - (2-2\sin\theta)^2\right]d\theta$$

$$A = 4\int_{\pi/6}^{\pi/4}\left(\cancel{\sin^2\theta} - 1 + 2\sin\theta - \cancel{\sin^2\theta}\right)d\theta = 4\Big|-\theta - 2\cos\theta\Big|_{\pi/6}^{\pi/2}$$

$$= 4\left[-\frac{\pi}{2} - \left(-\frac{\pi}{6} - 2\cdot\frac{\sqrt{3}}{2}\right)\right] = 4\left[-\frac{\pi}{2} + \frac{\pi}{6} + \sqrt{3}\right]$$

$$= 4\left[\frac{-3\pi+\pi}{6} + \sqrt{3}\right] = 4\left[\sqrt{3} - \frac{\pi}{3}\right]$$

Q7 $\quad 25x^2 + 4y^2 + 50x - 16y - 59 = 0$

$$25(x^2+2x) + 4(y^2-4y) - 59 = 0$$

$$25[x^2+2x+1-1] + 4[y^2-4y+4-4] - 59 = 0$$

$$25(x+1)^2 - 25 + 4(y-2)^2 - 16 - 59 = 0$$

$$\Rightarrow \quad 25(x+1)^2 + 4(y-2)^2 = 100$$

$$\Rightarrow \quad \frac{(x+1)^2}{4} + \frac{(y-2)^2}{25} = 1$$

$$\Rightarrow \quad \frac{(x-(-1))^2}{2^2} + \frac{(y-2)^2}{5^2} = 1$$

Ellipse with Major axis Parallel to y-axis

Center $(-1, 2)$ $\quad a = 5 \quad b = 2$

$b^2 = a^2 - c^2 \Rightarrow c^2 = a^2 - b^2$

$c^2 = 25 - 4 = 21 \quad c = \pm\sqrt{21}$

Vertices $(-1, 2\pm 5) = (-1, 7), (-1, -3)$

foci $(-1, 2\pm\sqrt{21}) = (-1, 2+\sqrt{21}), (-1, 2-\sqrt{21})$